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A VARIATIONAL SOLUTION OF SOLID AND FREE-FLOODING CYLINDRICAL SOUND RADIATORS OF FINITE LENGTH

Miguel C. Junger

1 March 1964

Technical Report U-177-48
Prepared for Office of Naval Research
Acoustics Programs - Code 468
Contract Nonr-2739(00)
Task NR 185-301

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ACKNOWLEDGMENT

An analytical study such as the one here presented is of practical value only if it can be readily used to obtain numerical results. This analysis escapes the fate of being a mere mathematical exercise because Dr. Joshua E. Greenspon, of J G Engineering Research Associates, Baltimore, Maryland, has applied to this analysis his vast experience in evaluating the complicated integrals characteristic of radiation problems in cylindrical coordinates. These quantitative results will be presented by Dr. Greenspon in a companion report: "Axially Symmetric Green's Functions for Cylinders." Drs. Alexander Silbiger and Ewald G. Eichler of this firm contributed useful comments.

ABSTRACT

The radiation impedance of a cylindrical sound source of finite length can be expressed as the sum of two components:

$$Z = Z_r + Z_\alpha$$

where Z_r is the impedance evaluated by means of the generally used technique originated by Robey, which assumes that the radiating surface is bracketed between two rigid semi-infinite cylindrical baffles. Z_α is associated with the radial velocity distribution $\alpha(z)$ over these two semi-infinite cylindrical surfaces and is therefore in the nature of a correction factor to Robey's impedance. $\alpha(z)$ is an unknown function which satisfies a non-homogeneous Fredholm integral equation. A functional $J[\alpha]$ is contructed which is stationary and proportional to Z_α for the correct solution $\alpha(z)$:

$$\delta J[\alpha]/\delta\alpha = 0, \quad Z_\alpha \propto J[\alpha]$$

From this variational principle a value of Z_α is calculated by means of a Rayleigh-Ritz-type procedure. Finally, the far field is evaluated. The variational principle used here parallels the Levine-Schwinger principle widely used to obtain scattering cross sections. Variational solutions are presented for solid and free-flooding cylinders for axisymmetric and for arbitrary velocity distributions. A variational solution is given for "squirters" of finite wall thickness, but it is restricted to thin walled transducers. In an Appendix, non-variational solutions of the integral equation for $\alpha(z)$ are presented for "squirters" of greater wall thickness.

In a companion report, the procedures developed here are applied to the evaluation of the self- and mutual-radiation impedances of elements in an array of coaxial, free-flooding axially spaced ring transducers. Dr. J. E. Greenspon, of J G Engineering Research Associates, evaluated the inverse Fourier transforms required for these solutions and obtained quantitative results which he will present in a separate report.

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Symbols*

(Alternative subscripts, viz $a_{\substack{o\\i}}$, are used to condense two equations into one, the upper subscript on the left side of the equation being associated with the upper signs and subscripts on the right side of the equation, and vice versa.)

a	radius of cylindrical source (Fig. 2); mean radius of "squirter" (Fig. 3)
a_i, a_o	inner and outer radius of "squirter," respectively (Fig. 3)
c	sound velocity in fluid medium
$G_i, G_o, g_i, g_o, G_{i+}, G_{i-}, \Gamma(z-z')$, and $H(z)$	Green's functions and related functions defined in table 1, p. 1
H_m	Hankel function of the first kind, of order m (with this notation, a massive reactance is negative)
h	half thickness of "squirter" (Fig. 3)
J_m	Bessel function of order m
k	wave number, equal to ω/c
k_r	radial wave number, equal to $(k^2-k_z^2)^{\frac{1}{2}}$
k_z	axial wave number
L	half length of cylindrical radiator (Figs. 2 and 3)
p	sound pressure
r,φ,z	cylindrical coordinates
R,θ	spherical coordinates
U	radial velocity amplitude of cylindrical source (Fig. 3)
$u(z)$	radial velocity distribution on cylindrical surface (r=a), positive outward (Fig. 3)
Z	radiation impedance of cylindrical source in units of force/velocity equal to (Z_r+Z_α) (Fig. 2)
Z_r	radiation impedance obtained from Robey's model (Fig. 1b) for which $\alpha(z) = 0$
Z_α	correction factor associated with velocity distribution $\alpha(z)$ and to be added to Z_r (Fig. 2)
$\alpha(z)$	radial velocity in the regions $\lvert z\rvert > L$ (Fig. 2) normalized to velocity amplitude U of cylindrical surface
ν	Poisson's ratio of transducer material
ρ	density of fluid medium
ϕ	velocity potential (outward velocity is $-\partial\phi/\partial r$); subscripts "i" and "o" refer respectively to regions $r \leq a$ and $r \geq a$
ω	circular frequency [harmonic time dependence factor $\exp(-i\omega t)$ which multiplies the velocities and the potentials, has been suppressed throughout this report]

*Other symbols are defined in the text.

Table 1

GREEN'S FUNCTIONS AND RELATED FUNCTIONS

Symbol	Equation	Region where Applicable	Sound Source Configuration to Which Applicable	Description
$G_o(r,a,z-z')$	IV.1	$r > a$	All configurations with axisymmetric velocity distribution	General form of axisymmetric Green's function
$G_i(r,a,z-z')$	IV.2	$r < a$	Free-flooding cylinder with axisymmetric velocity distribution	
$g_o(r,a,z-z')$	IV.6	$r > a$	All configurations with axisymmetric velocity distribution symmetrical about $z = 0$	Even component of G_o
$g_i(r,a,z-z')$	VIII.2	$r < a$	Free-flooding cylinder with axisymmetric velocity distribution symmetrical about $z = 0$	Even component of G_i
$G_{i+}(r,a; z-z')$	IV.3 and 4	$r < a$ $z > L$	Solid cylinder with axisymmetric velocity distribution	Green's function whose normal derivative $\partial/\partial z$ vanishes on end cap $z = L$
$G_{i-}(r,a; z-z')$	IV.3 and 4	$r < a$ $z < L$		Green's function whose normal derivative $\partial/\partial z$ vanishes on end cap $z = -L$
$G_{om}(r,a; \varphi-\varphi'; z-z')$	X.4	$r > a$	Cylinder with arbitrary velocity distribution expressed as a Fourier series in φ (Eq. X.[illegible])	General form of non-axisymmetric Green's function
$G_{im}(r,a; \varphi-\varphi'; z-z')$	[illegible].6	$r < a$		
$\Gamma(z-z')$	IV.11	$r = a$	Solid cylinder	Linear combination of Green's functions
	VIII.4a and VIII.5	$r = a$	Free-flooding cylinder of vanishing wall thickness	
	IX.14a	$r = a$	"Squirter"	
$\Pi(z)$	IV.8		Solid cylinder	Integral of Green's function over radiating surface
	VIII.4b		Free-flooding cylinder of vanishing wall thickness	
	IX.14b		"Squirter"	

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I. Scope of Study

In this report a variational technique is used to derive expressions for the radiation loading of radially pulsating cylinders of finite length. End effects are accounted for, no restrictions being placed on the circulation of the acoustic fluid around the edges of the cylinder. The far field potentials, on the axis of the cylindrical radiator and for other bearings are also given. The analysis is performed for (1) a solid cylinder, (2) an open-ended, free-flooding cylinder of vanishing wall thickness, and (3) a "squirter" of small, but non-vanishing thickness-to-radius ratio. The final section presents an extension of the analysis to cylindrical radiators embodying an arbitrary, non-axisymmetric velocity distribution. A non-variational technique presented in Appendix B extends this study to "squirters" of larger wall thickness.

With the present approach, the need for machine calculations has been confined to the evaluation of Green's functions in the form of Robey's integrals. As mentioned in the acknowledgment, these integrals are being evaluated by Dr. Greenspon, J G Engineering Research Associates. The technique developed in this study has been extended to the evaluation of the mutual radiation impedances between elements of an array of free-flooding ring transducers.* Numerical results for this configuration are also being obtained by Dr. Greenspon and will be included in his report: "Axially Symmetric Green's Functions for Cylinders."

II. A Review of Published Analytical Approaches to the Finite Cylindrical Radiator

The fluid potential Φ generated by a sound radiator is given by the familiar Helmholtz integral equation[1]

$$\Phi(\bar{R}) = \int_{S'} [G(\bar{R},\bar{R}') \frac{\partial\Phi(\bar{R}')}{\partial n'} - \frac{\partial G(\bar{R},\bar{R}')}{\partial n'} \Phi(\bar{R}')] \, dS' \qquad \text{(II.1)}$$

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*This analysis will be presented in a report to be published in March 1964, "Mutual Radiation Impedance for Spaced, Coaxial, Free-Flooding Ring Transducers," CAA Report U-178-48, Contract Nonr-2739.

where S' is the radiating surface, and n' is the outward normal to the surface of integration. The first term in the integrand can be readily evaluated, if the normal velocity $u(\overline{R}')$, of the radiating surface, which equals $-\partial\Phi/\partial n'$, is known. The second term in the integrand involves the unknown potential on the radiating surface, $\Phi(\overline{R}')$. We must therefore, in general, solve an integral equation to obtain this potential. In the last two years, the general availability of large, digital computers has made it practical to use a finite-difference method to obtain numerical solutions of the Helmholtz integral equation for the finite cylindrical radiator[2,3] (Fig. 1a and the Table on p.3). The drawback of this approach is that the large computational effort involved must be repeated for every combination of length-to-radius ratio, of ka, and of surface velocity distribution.

Another successful approach to the finite cylinder problem, which circumvents the Helmholtz integral equation, uses an expansion of the potential in spherical wave harmonics.[4] For this approach the volume of calculations is less than for the finite-difference approach described above. Thus, approximate results were obtained in ref. 4 without the help of electronic computers, by confining the series expansion to only a few terms. For practical applications, this method also requires computer facilities.

An approximate method which, historically, precedes the approaches described above, consists in constructing a Green's function whose derivative $\partial G/\partial n'$ vanishes on the infinite cylindrical surface $r=a$. If we now prolong the cylindrical radiator by two semi-infinite rigid cylindrical baffles of the same diameter, the surface integral in Eq. II.1 is confined to the cylindrical surface (Fig. 1b). Over this surface, the second term in the integrand, which involves the unknown potential, has been eliminated by our choice of the Green's function. We can therefore obtain an approximate expression for the potential without having to solve an integral equation:

$$\Phi(\overline{R}) = - \int_{S'} u(\overline{R}')G(\overline{R},\overline{R}')\, dS' \qquad (II.2)$$

With this approach Laird and Cohen[5] derived an analytical expression for the far field potential, the integrals being evaluated by the method of stationary phase. These integrals, which for the axisymmetric velocity distribution are known as Robey's integrals, must unfortunately be evaluated numerically if the potential on or near the radiating surface is required.[6] Greenspon has simplified the technique for performing this integration.[7] He and Sherman also evaluated these

COMPARISON OF VARIOUS ANALYTICAL APPROACHES TO THE CYLINDRICAL SOURCE OF FINITE LENGTH

	Salient Feature of Approach	Authors	Formulation of Problem	Electronic Computer Requirements
1	Finite-Difference Calculations (Fig. 1a)	Baron, Matthews and Bleich;[2] Chen and Schweikert[3]	Helmholtz Integral Equation over radiating surface using free-space Green's function, approximated by set of simultaneous algebraic finite-difference equations	Extensive
2	Series of Spherical Wave Harmonics	Parke and Williams[4]	Integral Equation Circumvented by expansion of potential in a series of spherical wave harmonics	Moderate
3	Semi-Infinite, Rigid Cylindrical Baffles Extending Radiator (Fig. 1b)	Laird and Cohen[5] for far field; Robey,[6] and Greenspon[7] and Sherman[8] for near field	Helmholtz Integral Equation circumvented by (1) constructing Green's function whose radial derivative vanishes on cylindrical surface, (2) extending radiator to infinity by rigid baffles	None for far field solution: moderate for radiation loading calculation ("Robey's integral")
4	Variational principle for radiation loading (Fig. 2)	Present report (Related to Schwinger's variational principle for scattering cross section)	Similar to approach (3), but unknown velocity distribution $\alpha(z)$ takes the place of semi-infinite baffles; $\alpha(z)$ satisfies integral equation; radiation impedance, stationary with respect to correct $\alpha(z)$, is solved by Rayleigh-Ritz-style calculation	Moderate (Same as "Robey's integral" evaluation in approach 3)

Table 2

integrals for non-axisymmetric velocity distributions.[8] The drawback of Robey's mathematical model is that it does not permit circulation of the fluid around the edges of the transducer, because of the assumption of two semi-infinite cylindrical baffles. Neither does this model lend itself to the evaluation of axially vibrating solid cylinders, or of the free-flooding open-ended cylinders known in acoustical vernacular as "squirters." Robey originally approximated the radiation loading of such free-flooding transducers by assuming that the fluid column inside the "squirter" is terminated by pressure-release pistons.[9] He then refined his analysis by assuming the terminal impedance to be that of a piston in an infinite plane baffle[10] (Fig. 1c).

In summary, existing analyses use either a large computational effort which must be repeated for every particular combination of sound source parameters, or an elegant approximate technique which, however, does not account for the circulation of fluid around the two extremities of the cylinder.

III. Description of the Present Approach

The present approach makes use of a Green's function similar to Robey's, thus eliminating from the Helmholtz equation the term containing the potential on the cylindrical surface $r=a$. However, instead of assuming the source to be bracketed by rigid baffles, the potential is expressed in terms of an unknown radial velocity distribution, $\alpha(z)$, over the two semi-infinite cylindrical boundaries prolonging the sound source (Fig. 2). The potential in the cylindrical column in the region $r < a$ is then formulated with the help of a suitable Green's function whose normal derivative vanishes on the boundary $r=a$. The potential in this cylindrical region is also expressed in terms of the unknown velocity distribution $\alpha(z)$. By requiring continuity of these two potentials across the cylindrical boundary $r=a$, $|z| > L$, an integral equation for $\alpha(z)$ is obtained. If we compare this formulation to the free-space Green's function formulation in ref. 2 and 3, we see that the unknown function $\alpha(z)$ and the surface $r=a$, $|z| > L$ take, respectively, the place of $\Phi(\overline{R}')$ and of the radiator surface. One advantage of the present approach is that a given error in the expression for $\alpha(z)$ can be expected to result in a smaller error in the radiation impedance than would result from a similar error in $\Phi(\overline{R}')$ in ref. 2 and 3. The principal advantage, however, is that this approach lends itself to an approximate variational solution both of

the solid and open-ended finite cylinder.*

The radiation impedance Z can be written as the sum of the impedance Z_r obtained by setting $\alpha(z) = 0$, and of an impedance Z_α associated with $\alpha(z)$, the unknown velocity distribution in the region $|z| > L$:

$$Z = Z_r + Z_\alpha \tag{III.1}$$

Z_α is thus in the nature of a correction factor to impedances computed by Robey,[6] Greenspon,[7,8] and Sherman[8] from Robey's mathematical model. By virtue of the variational principle to be derived in Section V, for the correct solution of the integral equation $\alpha(z)$, Z_α is proportional to a functional $J[\alpha]$ which is stationary with respect to first order variations of $\alpha(z)$:

$$Z_\alpha \propto J[\alpha]$$

$$\frac{\delta J[\alpha]}{\delta \alpha} = 0 \tag{III.2}$$

Furthermore, $J[\alpha]$ depends on the functional form of $\alpha(z)$ but not on its amplitude. The technique for computing Z_α is similar to the Rayleigh-Ritz technique for evaluating the natural frequency.

This approach parallels the use of the Levine-Schwinger variational principle for scattering cross sections, which has been applied to a large number of diffraction problems.[12] The equivalent principle for radiation impedances is proved in its general form, using free-space Green's functions, by Morse and Feshbach.[13] These authors do not, however, use it to solve any particular problem. Apparently, only Storer[14] applied this principle to a specific problem, viz. the effect of a finite circular baffle on the radiation loading of a coaxial antenna. In 1954, Professor Storer, of Harvard University, suggested to the author of this report that the axisymmetrically vibrating cylinder of finite length could also be analyzed in this fashion. Consequently a rather sketchy variational solution

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*A rigorous Wiener-Hopf type solution of the integral equation is possible for semi-infinite cylindrical radiator problems formulated in this fashion, according to Levine and Schwinger.[11] These authors mention this formulation as an alternative to the one they actually used in their analysis of sound radiation from a semi-infinite pipe. Levine,[12a] extended this study to pipes of arbitrary cross section. One of his approximations for the reflection coefficient is obtained from a variational solution of an integral equation (his "variational principle A") appliable over the semi-infinite cylindrical surface extending the pipe, and is therefore of the form of the integral equation used in this report.

of the solid cylindrical source was presented in an internal memorandum of the Harvard Acoustics Research Laboratory.[15] A detailed analysis was not carried out because numerical results depended on the evaluation of Robey's integral, which at that time was not available. Since, as mentioned earlier, such integrals can now be readily evaluated, and since Dr. Greenspon kindly agreed to apply his experience in this type of calculation to the problem at hand, it is now worthwhile to use the variational formulation to obtain a solution to the finite cylinder problem.

IV. Integral Equation Formulation of the Solid Cylinder Problem

The infinite region surrounding the cylindrical radiator is subdivided into three regions (Fig. 2): an outer region, $r > a$, identified by subscript o; and two semi-infinite inner regions, $r < a$, one corresponding $z > L$ and identified by the subscript i+, and a second inner region corresponding to values of $z < L$, identified by the subscript i-. The Green's function for the outer region satisfying the condition $\partial G/\partial r'=0$ for $r'=a$, was constructed by Robey:[6]

$$G_o(r,a,z-z') = \frac{1}{4\pi^2 a} \int_{-\infty}^{\infty} \frac{H_o(k_r r)}{k_r H_1(k_r a)} \exp[ik_z(z-z')]\, dk_z \tag{IV.1}$$

The evaluation of G_o is the subject of references 6 and 7. Since $k_r=(k^2-k_z^2)^{\frac{1}{2}}$, the functions of k_r in the integrand are even in k_z. Hence, only the real component of the exponential function, $\cos[k_z(z-z')]$ contributes to the integral. The same comment applies to the Green's function for the infinite cylindrical region $r < a$:

$$G_i(r,a,z-z') = -\frac{1}{4\pi^2 a} \int_{-\infty}^{\infty} \frac{J_o(k_r r)}{k_r J_1(k_r a)} \exp[ik_z(z-z')]\, dk_z \tag{IV.2}$$

This function is derived and evaluated in Appendix A.

In the case of the solid cylindrical radiator it is convenient to combine two Green's functions of the form of Eq. IV.2 so that the normal derivative of the resultant Green's function vanishes over the end caps of the cylinder, i.e., in the two circular regions $r < a$, $z = \pm L$. This will eliminate the potential term from the Helmholtz integral over the end caps, as well as over the cylindrical surface.

Such Green's functions are readily constructed by introducing image sources:

$$G_{i+}(r,a,z-z') = G_i(r,a,z-z') + G_i(r,a,z+z'-2L)$$

$$G_{i-}(r,a,z-z') = G_i(r,a,z-z') + G_i(r,a,z+z'+2L) \qquad \text{(IV.3)}$$

These Green's functions can be written more concisely as*

$$G_{i\pm}(r,a,z-z') = -\frac{1}{2\pi^2 a}\int_{-\infty}^{\infty} \frac{J_0(k_r r)}{k_r J_1(k_r a)} \cos[k_z(z\mp L)]\cos[k_z(z'\mp L)]\,dk_z \qquad \text{(IV.4)}$$

We can now write the potentials in these three regions by making use of the modified Helmholtz integral in Eq. II.2. If we assume that the end caps are rigid, the surface integral reduces to the cylindrical surface:**

$$\Phi_o(r,z) = 2\pi a\int_{-\infty}^{\infty} u(z')G_o(r,a,z-z')\,dz' \qquad \text{(IV.5a)}$$

$$\Phi_{i\pm}(r,z) = \mp 2\pi a\int_{\pm L}^{\pm\infty} u(z')[G_i(r,a,z-z') + G_i(r,a,z+z' \mp 2L)]\,dz' \qquad \text{(IV.5b)}$$

The time dependence of $u(z')$ and of the potentials is harmonic. The time-dependent function $\exp(-i\omega t)$ has been omitted, for the sake of brevity throughout this report. These integrals are of opposite sign, because $\partial\Phi/\partial n'$ in the Helmholtz equation equals $-\partial\Phi_o/\partial r = u$ in the outer region, and $+\partial\Phi_i/\partial r = -u$ in the inner region. With this sign convention, the sound pressure equals $\rho\dot{\Phi}$. If the velocity distribution of the radiator is symmetrical about the plane $z=0$, the two inner regions will have identical potentials and we need concern ourselves with only one inner region which we will designate by the subscript i. For the case of a symmetrical velocity distribution, only the part of the Green's function which is symmetrical

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*Here, and elsewhere in this report, alternative subscripts and signs have been used, for the sake of brevity, to condense two equations into one, the upper subscript on the left side of the equation being associated with the upper sign on the right side of the equation, and vice versa.

**In section X expressions are given for an arbitrary velocity distribution over the radiating surface. The potential contributed by vibrating end caps is given in Eq. X.1 and 8.

about $z'=0$ contributes to the potential. We will denote this even component of G_o by g_o:

$$g_o(r,a,z-z') = \frac{1}{2\pi^2 a} \int_0^\infty \frac{H_o(k_r r)}{k_r H_1(k_r a)} \cos k_z z \cos k_z z' \, dk_z \tag{IV.6}$$

We further specialize the problem by assuming that the radial velocity of the sound source is constant and equal to U. $u(z')$ is therefore a known function for $|z| < L$. The velocity distribution along the cylindrical boundaries $z > L$, $r=a$, is an unknown function, say $U\alpha(z)$. The integrals for the potentials now become

$$\Phi_1(r,z) = -2\pi a \, U \int_L^\infty \alpha(z')[G_1(r,a,z-z') + G_1(r,a,z+z'-2L)] \, dz' \tag{IV.7a}$$

$$\Phi_o(r,z) = 4\pi a \, U[\int_0^L g_o(r,a,z-z') \, dz' + \int_L^\infty \alpha(z') g_o(r,a,z-z') \, dz'] \tag{IV.7b}$$

For the sake of brevity we will from now on express the known component of the potential Φ_o, evaluated on the cylindrical surface $r=a$, as a function, say $H(z)$, rather than as an integral

$$H(z) = \int_0^L g_o(a,a,z-z') \, dz' \tag{IV.8}$$

We can now construct the integral equation which the unknown function $\alpha(z)$ must satisfy. This integral equation is derived from the requirement that the potentials be continuous across the cylinder boundary $r=a$, $z > L$:

$$\Phi_o(a,z) - \Phi_1(a,z) = 0, \quad \text{for } z > L \tag{IV.9}$$

When we substitute Eqs. IV.5, this continuity condition takes the form of a non-homogeneous Fredholm integral equation of the first kind

$$\int_L^\infty \Gamma(z-z') \, \alpha(z') \, dz' = -H(z), \quad \text{for } z > L \tag{IV.10}$$

where the kernel of this equation is given by

$$I(z-z') = g_o(a,a,z-z') + \frac{1}{2} G_1(a,a,z-z') + \frac{1}{2} G_1(a,a,z+z'-2L) \qquad \text{(IV.11)}$$

We will now show that the function $\alpha(z')$ which satisfies this integral equation gives a stationary value for the radiation impedance, with respect to variations $\delta\alpha$.

V. Derivation of the Variational Principle for the Radiation Impedance

We note for future use that the resultant radiation impedance on the cylinder is obtained by integrating the pressure

$$\begin{aligned} p(a,z) &= \rho\dot{\Phi}_o(a,z) \\ &= -i\omega\rho\Phi_o(a,z) \qquad \text{(V.1)} \end{aligned}$$

over the surface of the cylinder:

$$\begin{aligned} Z &= -\frac{4i\pi a\omega\rho}{U} \int_0^L \Phi_o(a,z)\, dz \\ &= -i(4\pi a)^2\omega\rho \int_0^L [H(z) + \int_L^\infty \alpha(z')g_o(a,a,z-z')\, dz']\, dz \qquad \text{(V.2)} \end{aligned}$$

The radiation impedance is thus clearly the sum of two component impedances, as indicated in Eq. III.1. The impedance computed from Robey's model is

$$\begin{aligned} Z_r &= -(4\pi a)^2\, i\omega\rho \int_0^L H(z)\, dz \\ &= -(4\pi a)^2\, i\omega\rho \int_0^L\int_0^L g_o(a,a,z-z')\, dz'\, dz \qquad \text{(V.3)} \end{aligned}$$

Like $H(z)$, Z_r is a known quantity since it does not involve the unknown function $\alpha(z)$. The correction term in Eq. III.1, which embodies the contribution of the

flow across the cylindrical boundaries prolonging the source is

$$Z_\alpha = -(4\pi a)^2 \, i\omega\rho \int_0^L [\int_L^\infty \alpha(z') \, g_o(a,a,z-z') \, dz'] \, dz \qquad (V.4)$$

Since the Green's function is symmetrical in z and z' the order of integration in Eq. V.4 can be inverted:

$$Z_\alpha = -(4\pi a)^2 \, i\omega\rho \int_L^\infty [\int_0^L g_o(a,a,z-z') \, dz'] \, \alpha(z) \, dz$$

$$= -(4\pi a)^2 \, i\omega\rho \int_L^\infty H(z) \, \alpha(z) \, dz \qquad (V.5)$$

A functional $J[\alpha]$ will be defined below, Eqs. V.9 and 10. For future reference we note that for the correct function $\alpha(z)$, i.e., for the function which satisfies the integral equation, Eq. IV.10, this functional can also be written as

$$J[\alpha] = -\int_L^\infty H(z) \, \alpha(z) \, dz, \quad \text{for } \alpha(z) \text{ solution of Eq. IV.10.} \qquad (V.6)$$

Comparing this with Eq. V.5, we can relate this functional and the impedance Z_α:

$$Z_\alpha = i\omega\rho(4\pi a)^2 J[\alpha], \quad \text{for } \alpha(z) \text{ solution of Eq. IV.10.} \qquad (V.7)$$

We will now prove that $J[\alpha]$ is stationary with respect to first order variations $\delta\alpha$ about the correct function $\alpha(z)$. For this purpose, we relate $J[\alpha]$ to the integral equation, Eq. IV.10. We multiply both sides of this equation by $\alpha(z)$ and integrate with respect to z over the region $z > L$. We then divide both sides of the equation thus obtained by

$$[\int_L^\infty H(z) \, \alpha(z) \, dz]^2$$

Our original integral equation now takes the form

$$\frac{\int_L^\infty \int_L^\infty \alpha(z)\Gamma(z-z')\,\alpha(z')\,dz'\,dz}{\left[\int_L^\infty H(z)\alpha(z)\,dz\right]^2} = \frac{-1}{\int_L^\infty H(z)\alpha(z)\,dz} \tag{V.8}$$

The functional $J[\alpha]$ is defined as the reciprocal of the left side of this equation:

$$J[\alpha] = \frac{A^2[\alpha]}{B[\alpha]} \tag{V.9}$$

where

$$A[\alpha] = \int_L^\infty H(z)\,\alpha(z)\,dz \tag{V.10a}$$

$$B[\alpha] = \int_L^\infty \int_L^\infty \alpha(z)\,\Gamma(z-z')\,\alpha(z')\,dz'\,dz \tag{V.10b}$$

For a function $\alpha(z)$ which satisfies the integral equation, and therefore the equality in Eq. V.8, the reciprocal of the right side of Eq. V.8 is also equal to the $J[\alpha]$, as already indicated in Eq. V.6. If the functional defined in Eq. V.9 is indeed stationary with respect to small variations of the function $\alpha(z)$, then, by definition, the increment $\delta J[\alpha]$ associated with an increment $\delta\alpha$ is zero:

$$\delta J[\alpha] = \frac{2A[\alpha]}{B[\alpha]} \cdot \int_L^\infty H(z)\,\delta\alpha(z)\,dz -$$

$$- \frac{A^2[\alpha]}{B^2[\alpha]} \cdot \int_L^\infty \int_L^\infty \Gamma(z-z')[\alpha(z)\delta\alpha(z')+\alpha(z')\delta\alpha(z)]\,dz'\,dz = 0 \tag{V.11}$$

Like the Green's functions in Eq. IV.11, $\Gamma(z-z')$ is symmetrical with respect to z and z'. We can therefore invert the order of integration in the former of the two terms of the integrand of the double integral in Eq. V.11. The double integral can thus be condensed to

$$\int\int = 2 \int_L^\infty \int_L^\infty \Gamma(z-z')\,\alpha(z')\delta\alpha(z)\,dz'\,dz \tag{V.12}$$

From the integral equation, Eq. IV.10, we see that the integral over z' equals $-H(z)$ for the correct function $\alpha(z)$. The double integral can thus finally be written as

$$\iint = -2\int_L^\infty H(z)\ \delta\alpha(z)\ dz \tag{V.13}$$

When we substitute Eq. V.13 in place of the double integral in Eq. V.11 and multiply the terms of this equation by the ratio $B^2[\alpha]/2A[\alpha]$, we obtain

$$\{B[\alpha] + A[\alpha]\} \cdot \int_L^\infty H(z)\ \delta\alpha(z)\ dz = 0 \tag{V.14}$$

Since the integral is not identically zero the sum in brackets, which multiplies this integral, must vanish, i.e.,

$$B[\alpha] = -A[\alpha] \tag{V.15}$$

When we substitute the definitions of these two functionals, Eqs. V.10, this becomes:

$$\int_L^\infty\int_L^\infty \alpha(z)\ \Gamma(z-z')\ \alpha(z')\ dz'\ dz = -\int_L^\infty H(z)\ \alpha(z)\ dz \tag{V.16}$$

This equation is obviously satisfied if $\alpha(z')$ satisfies our original integral equation, Eq. IV.10. We have thus shown that the functional $J[\alpha]$, as defined in Eq. V.9, does indeed take on a stationary value for the correct value of the function $\alpha(z)$. Since the functional $J[\alpha]$ is stationary with respect to the correct function, the error in $J[\alpha]$ is of a higher order than the error in $\alpha(z)$.

We shall now illustrate the evaluation of the radiation impedance by means of the variational principle just derived.

VI. Evaluation of the Radiation Impedance from the Variational Principle

We first proceed to select the simplest trial function $\alpha(z)$ which yields a far field potential Φ_1 in the desired form of a spherically spreading wave,

$$\Phi_1(a,z) = \frac{A}{|z|} \exp(ikz) \ , \quad \text{for large } |z| \tag{VI.1}$$

If the cylindrical boundary r=a, $|z| > L$ were in the form of a rigid pipe, the far field potential would only decay as a result of viscous losses, as embodied in an imaginary component of the wave number. The far field potential in a rigid pipe can therefore only decay exponentially. The desired spherical spreading loss, Eq. VI.1, must therefore be the result of energy flow across the cylindrical boundary associated with the velocity $\alpha(z)$. The rate of energy outflow, per unit axial distance, along the "pipe" is

$$\frac{\partial \dot{E}(z)}{\partial z} = \pi k a \rho c U \left|\Phi_1(a,z)\right| \left|\alpha(z)\right| , \tag{VI.2}$$

This must balance the decrease, per unit axial distance, of the acoustic energy propagating down the pipe:

$$\frac{\partial \dot{E}(z)}{\partial z} = \frac{\partial}{\partial z} \left[\pi \rho c k^2 \int_0^a \left|\Phi_1(r,z)\right|^2 r dr\right] \tag{VI.3}$$

In the far field, and for the values of kL characteristic of transducers, $\Phi_1(r,z)$ can be set equal to $\Phi_1(a,z)$.* We can thus replace the integral in Eq. VI.3 by $a^2|\Phi_1(a,z)|^2/2$. When we substitute Eq. VI.1 for Φ_1 in Eqs. VI.2 and 3, we can solve for $|\alpha(z)|$:

$$|\alpha(z)| = \frac{kaA}{U} \frac{1}{|z|^2} \ , \quad \text{for large } |z| \tag{VI.4}$$

We thus conclude, that in the far field, $\alpha(z)$ must be of the form

$$\alpha(z) \propto \frac{\exp(ikz)}{|z|^2} \ , \quad \text{for large } |z| \tag{VI.5}$$

We will now illustrate the use of the variational principle by selecting the

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*Even for large ka, the value of Φ_1 averaged over the cylindrical cross section is equal to a constant times $\Phi_1(a,z)$. The functional relation derived in Eq. VI.4 therefore still holds, but the constant in this equation will not equal A.

simplest trial function which satisfies Eq. VI.5 and which can also account for a more rapidly decaying near field. Such a function requires the use of at least two unknown coefficients, x_1 and x_2:

$$\alpha(z) = (\frac{x_1}{z^2} + \frac{x_2}{z^3}) \exp(ikz) \qquad \text{(VI.6)}$$

Since x_1 and x_2 are generally complex quantities, this expression allows for a phase shift between the two components of the potentials. By virtue of the variational principle, the best values of the unknown coefficients are those which give a stationary value to the functional $J[\alpha]$:

$$\partial J[\alpha]/\partial x_1 = 0$$

$$\partial J[\alpha]/\partial x_2 = 0 \qquad \text{(VI.7)}$$

When we substitute Eq. V.9 for the functional $J[\alpha]$ we can write these equations, after some manipulation, as

$$2\,\frac{\partial A[\alpha]}{\partial x_1} A[\alpha] - \frac{\partial B[\alpha]}{\partial x_1} J[\alpha] = 0 \qquad \text{(VI.8)}$$

etc.

When we combine the assumed trial function, Eq. VI.6, with the definitions of the functionals $A[\alpha]$ and $B[\alpha]$, Eqs. V.10, these two functionals are found to be, respectively, linear and quadratic in x_1 and x_2,

$$A[\alpha] = a_1 x_1 + a_2 x_2, \quad \partial A/\partial x_1 = a_1, \quad \partial A/\partial x_2 = a_2$$

$$B[\alpha] = b_1 x_1^2 + b_2 x_2^2 + 2b_{12}\, x_1 x_2$$

$$\partial B/\partial x_1 = 2b_1 x_1 + 2b_{12} x_2, \quad \partial B/\partial x_2 = 2b_2 x_2 + 2b_{12}\, x_1 \qquad \text{(VI.9)}$$

where coefficients a_1, a_2, b_1, b_2 and b_{12} are known, complex quantities. When expressions VI.9 are substituted in Eqs. VI.8 a set of two simultaneous linear equations is obtained:

$$2(a_1^2 - b_1\, J[\alpha])x_1 + 2(a_1 a_2 - b_{12}\, J[\alpha])x_2 = 0$$

$$2(a_1 a_2 - b_{12}\, J[\alpha])x_1 + 2(a_2^2 - b_2\, J[\alpha])x_2 = 0 \qquad \text{(VI.10)}$$

Since this set of equations is homogeneous, its coefficient matrix must vanish:

$$\begin{vmatrix} 2(a_1^2-b_1J[\alpha]) & 2(a_1a_2-b_{12}J[\alpha]) \\ 2(a_1a_2-b_{12}J[\alpha]) & 2(a_2^2-b_2J[\alpha]) \end{vmatrix} = 0 \tag{VI.11}$$

Expanding this matrix we only obtain terms proportional to $J^2[\alpha]$ and $J[\alpha]$. The equations can therefore be solved for $J[\alpha]$:

$$J[\alpha] = \frac{a_1^2b_2 + a_2^2b_1 - 2a_1a_2b_{12}}{b_1b_2 - b_{12}^2} \tag{VI.12}$$

The remarkable feature of this result is that the functional is independent of x_1 and x_2. Like the coefficients a_1, a_2, b_1, b_2 and b_{12} the functional in Eq. VI.12 is complex. We see, by referring to Eq. V.7, that the imaginary component of $J[\alpha]$ embodies the radiation resistance, and its real component, the reactance.

If we are merely interested in the radiation loading we need not evaluate the unknown coefficients x_1 and x_2. If, however, we wish to compute the far field potentials, we must substitute the expression for $\alpha(z)$ in Eqs. IV.8, and therefore require the values of x_1 and x_2. The ratio of these two coefficients is obtained from either of the two homogeneous equations, Eq. VI.10

$$\frac{x_2}{x_1} = \frac{(-a_1^2 + b_1J[\alpha])}{a_1a_2 - b_{12}J[\alpha]} \tag{VI.13}$$

where the value of $J[\alpha]$ is known from Eq. VI.12. The coefficient x_1 is obtained by substituting the ratios x_2/x_1, Eq. VI.13 in Eq. VI.6, which is then substituted for $\alpha(z')$ in the integral equation, Eq. IV.10. Unless the functional dependence of the trial function, Eq. VI.6, on z is the correct one, the coefficient x_1 can not be selected so as to satisfy the integral equation in the whole range $|z| > L$. It is advantageous to select a coefficient x_1 which satisfies the integral equation for a value of z associated with a relatively large value of $\alpha(z)$ and hence with a large contribution to the far field potential, viz. for $z \approx L$, say $L + \epsilon$:

$$x_1 = \frac{-H(L+\epsilon)}{\int_L^\infty \Gamma(L+\epsilon-z') \cdot \left(\frac{1}{z'^2} + \frac{x_2}{x_1}\frac{1}{z'^3}\right) \exp(ikz')\, dz'} \tag{VI.14}$$

An alternative procedure, which gives more nearly equal weight to the whole region of z where the integral equation applies, is based on the fact that for the correct function $\alpha(z)$, $J[\alpha]$ equals $-A[\alpha]$, from Eqs. V.6 and V.10a. Hence, substituting the ratio x_2/x_1, from Eq. VI.13, and the value of $J[\alpha]$ from Eq. VI.12, in the expression for $A[\alpha]$, Eq. VI.9, we can solve for x_1

$$x_1 = \frac{-J[\alpha]}{a_1+a_2(x_2/x_1)} \tag{VI.15}$$

Experience with numerical calculations will indicate which procedure is preferable.

To refine the selection of the trial function further, we can introduce additional unknown coefficients associated, for example, with non-propagating incompressible near field components of the potentials. The trial function might thus, for example, be expressed in terms of three coefficients: x_1 and x_2, associated with propagating components of the potentials, and x_3 with an incompressible, near-field component decaying rapidly with distance:

$$\alpha(z) = \left(\frac{x_1}{z^2} + \frac{x_2}{z^3}\right) \exp(ikz) + \frac{x_3}{z^3} \tag{VI.6a}$$

This yields three simultaneous equations of the form of Eq. VI.8. Once again, we will find that these equations are linear in the three unknown coefficients and, of course, homogeneous. We can therefore construct a third order determinant similar to Eq. VI.11. The constant term and the linear term in $J[\alpha]$ are found to cancel, leaving only a cubic and a quadratic term in $J[\alpha]$. The determinant thus yields a single root $J[\alpha]$:

$$\begin{aligned} J[\alpha] = [&a_1^2(b_2b_3-b_{23}^2) + a_2^2(b_1b_3-b_{13}^2) + a_3^2(b_1b_2-b_{12}^2) + \\ &+ 2a_1a_2(b_{13}b_{23}-b_3b_{12}) + 2\,a_1a_3(b_{12}b_{23}-b_2b_{13}) \\ &+ 2a_2a_3(b_{12}b_{13}-b_1b_{23})] \cdot [b_1b_2b_3+2b_{12}b_{13}b_{23} \\ &- (b_1b_{23}^2+b_2b_{13}^2+b_3b_{12}^2)]^{-1} \end{aligned} \tag{VI.12a}$$

We then solve three of the set of three homogeneous equations, Eqs. VI.8, for two ratios of underdetermined coefficients. Finally, we solve for the amplitude of the one remaining coefficient by satisfying the integral equation at $z = L + \epsilon$, or in the manner indicated in Eq. VI.15.

If $\alpha(z)$ is expressed in terms of N unknown coefficients, the functionals $A[\alpha]$ and $B[\alpha]$ and their derivatives take on the following form:

$$A[\alpha] = \sum_{n=1}^{N} a_n x_n \ , \quad \frac{\partial A}{\partial x_n} = a_n$$

$$B[\alpha] = \sum_{n=1}^{N} \left(b_n x_n^2 + 2\sum_{m\neq n}^{N} b_{nm} x_n x_m\right)$$

$$\frac{\partial B}{\partial x_n} = 2b_n x_n + 2\sum_{m\neq n}^{N} b_{nm} x_m \qquad \text{(VI.9a)}$$

The set of N homogeneous linear equations for the unknown coefficients corresponding to Eqs. VI.10 is of the general form

$$2(a_n^2 - b_n J[\alpha])x_n + 2\sum_{m\neq n}^{N} (a_n a_m - b_{nm} J[\alpha])x_m = 0 \qquad \text{(VI.10a)}$$

When the coefficient matrix of this set of equations is set equal to zero it will be found that only the terms containing the two highest powers of $J[\alpha]$, N and N-1, do not cancel. When both terms are divided by $(J[\alpha])^{N-1}$, a linear equation in $J[\alpha]$ is obtained. The Nth order determinant for $J[\alpha]$ has a single non-vanishing root. This is consistent with the requirement that the integral equation, Eq. IV.10, have only one solution.

Experience with numerical calculations will show whether the radiation impedance is sensitive to the selection of the trial function $\alpha(z)$. If this should be the case, the functional dependence of the near field potentials on z, particularly of the non-propagating incompressible components, can be studied more closely so as to construct a more sophisticated trial function than Eqs. VI.6 or 6a. Theoretical insight into this functional relation can be gained from the fluid mechanics literature dealing with accessions to inertia of vibrating solids. Comparison with the results of the non-variational solutions presented in Appendix B can also be used to evolve more refined expressions of $\alpha(z)$.

The fact that $J[\alpha]$ (or, referring to Eq. V.7, Z_α) can be evaluated from a variational principle, without previously determining the amplitude of the

unknown coefficients of the trial function $\alpha(z)$, has already been related to the Levine-Schwinger variational principle for scattering cross sections (Section III). Another parallel which may be more familiar to some readers is found in the Rayleigh-Ritz method for optimizing the natural frequencies obtained from Rayleigh's principle.[16] In this method a trial function is assumed for the dynamic configuration of the vibrating system. The best choice of the coefficients in this trial function is determined by giving a stationary value to the natural frequency obtained from Rayleigh's principle. If N unknown coefficients are used in expressing the trial function, a set of N linear homogeneous equations is obtained with the coefficients as unknown quantities. By setting the coefficient matrix of this set of equations equal to zero, values of natural frequencies are obtained, without ever having to compute the unknown coefficients themselves. The fundamental natural frequency* thus obtained is equivalent to the functional $J[\alpha]$. To compute the ratio of the undetermined coefficients at that frequency one substitutes this value of the fundamental frequency back in the set of N homogeneous equations and solves for N-1 ratios. The amplitude of the Nth unknown coefficient is finally obtained from an inhomogeneous equation of motion.

To conclude our study of the solid cylindrical radiator, we now turn to the evaluation of the far field potentials.

VII. The Far Field Potentials

To evaluate the potential in the region $r < a$, $z > L$ we substitute the Green's function, G_1, Eq. IV.2, in Eq. IV.7a:

$$\phi_1(r,z) = \frac{U}{2\pi}\int_L^\infty \alpha(z')dz' \int_{-\infty}^{\infty} \frac{J_0(k_r r)}{k_r J_1(k_r a)} \{\exp[ik_z(z-z')]+\exp[ik_z(z+z'-2L)]\}dk_z \quad \text{(VII.1)}$$

For $\alpha(z')$, we substitute a trial function of the form of Eq. VI.6, with the unknown coefficients expressed in terms of $J[\alpha]$, as described in Section VI. The k_z-integral associated with the first exponential term in braces in Eq. VII.1 is

*In contrast to the variational principle used here, which yields a single solution $J[\alpha]$, the Rayleigh-Ritz techniques yields a number of natural frequencies equal to the number of unknown coefficients in the assumed trial function.

is given in Appendix A, Eq. A.14. Substituting $(-z'+2L)$ in place of z', we obtain the k_z-integral associated with the second exponential term. When we add the two integrals, we obtain

$$\int_{-\infty}^{\infty} dk_z = 2\pi i\left[\frac{\exp(ik|z-z'|)}{ka} + \frac{\exp[ik(z+z'-2L)]}{ka}\right] + 4\pi i \sum_{n=1}^{\infty} \frac{J_o(k_n r)}{(k^2-k_n^2)^{\frac{1}{2}}\, a[J_o(k_n a)-J_2(k_n a)]} \cdot$$

$$\{\exp[i(k^2-k_n^2)^{\frac{1}{2}}|z-z'|] + \exp[i(k^2-k_n^2)^{\frac{1}{2}}(z+z'-2L)]\} \qquad \text{(VII.2)}$$

where $(k_n a)$ is the nth root of the Bessel function J_1. The integral over z' in Eq. VII.1, must be split into two regions of integration: (1) from L to z, where $|z-z'|$ is taken equal to $(z-z')$, and (2) from z to ∞, where $|z-z'|$ equals $-(z-z')$. Since even the lowest root, $k_1 a=3.83$, is generally larger than the ka-value of resonant piezoelectric or magnetostrictive transducers, the terms under the summation sign decay exponentially with increasing $|z-z'|$. Because the source distribution $\alpha(z)$ extends to infinity, these "near field" terms contribute to the far field. By using energy flow considerations it was shown in Section VI that the desired far field behavior of ϕ_i, Eq. VI.1, requires that the function $\alpha(z)$ embody terms of order $|z|^{-2}$ and higher. The dominant, plane-wave components of the inverse transform of the Green's function, Eq. VII.2, do not decay with increasing z'. In combination with the far field term of $\alpha(z')$, these plane wave components of the Green's function therefore give rise to a far field potential whose absolute value varies as $\int |z|^{-2}dz = -|z|^{-1}$. This result is consistent with the potential, Eq. VI.1, used in deriving the functional form of $\alpha(z)$ in the far field. The evaluation of the inverse transform of the Green's function, Eq. A.14, is thus consistent with the energy flow analysis in Eqs. VI.2 to 4.

The far field in the region $r > a$ is obtained by substituting the appropriate Green's function, Eq. IV.1, in Eq. IV.5a:

$$\phi_o(r,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{H_o(k_r r)}{k_r H_1(k_r a)} \exp[ik_z(z-z')]\, u(z')\, dk_z\, dz' \qquad \text{(VII.3)}$$

The integration over z' can be carried out immediately by making use of the definition of the Fourier transform of the velocity distribution $u(z')$:

$$u(k_z) = \int_{-\infty}^{\infty} u(z') \exp(-ik_z z')\, dz' \qquad \text{(VII.4)}$$

The expression for the potential now becomes

$$\Phi_o(r,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{u(k_z)H_o(k_r r)}{k_r H_1(k_r a)} \exp(ik_z z)\, dk_z \tag{VII.5}$$

When the asymptotic, large argument expression for the Hankel function,

$$H_o(k_r r) = \left(\frac{2}{\pi k_r r}\right)^{\frac{1}{2}} \exp[i(k_r r - \frac{\pi}{4})] \tag{VII.6}$$

is substituted in the integrand in Eq. VII.5, and using spherical coordinates R and θ in lieu of the cylinder coordinates,

$$z = R\cos\theta \quad \text{and} \quad r = R\sin\theta,$$

the far field potential becomes

$$\Phi_o(R,\theta) = \frac{\exp(-i\pi/4)}{(2\pi^3 R\sin\theta)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{\exp[iR(k_r\sin\theta + k_z\cos\theta)]}{k_r^{\frac{3}{2}} H_1(k_r a)}\, dk_z \tag{VII.7}$$

This integral was evaluated by Laird and Cohen[5] using the method of stationary phase. The result thus obtained is

$$\Phi_o(R,\theta) = \frac{-iu(k\cos\theta)\exp(ikR)}{\pi kR\sin\theta\, H_1(ka\sin\theta)} \tag{VII.8}$$

The velocity transform $u(k\cos\theta)$ can be written more explicitly in terms of the trial function $\alpha(z')$ as

$$u(k\cos\theta) = 2U\left[\frac{\sin(kL\cos\theta)}{k\cos\theta} + \int_L^{\infty}\alpha(z')\cos(kz'\cos\theta)\, dz'\right] \tag{VII.9}$$

We now have concluded the analysis of the solid cylindrical radiator, and proceed with the open-ended cylinder.

VIII. The Open-Ended Free-Flooding Cylindrical Radiator of Vanishing Wall Thickness

We consider a cylindrical radiator whose wall thickness is negligible compared to both its radius and the acoustic wavelength. For such a source configuration we can construct a single potential ϕ_i for the region $r < a$ extending now from $-\infty < z < \infty$. This is in contrast to the solid radiator where we had to define two potentials ϕ_{i+} and ϕ_{i-} each valid in a semi-infinite region. The outer potential ϕ_o is similar to the outer potential derived for the solid cylinder, Eqs. IV.5a and 7a. The inner potential is of the form

$$\phi_i(r,z) = -2\pi a \int_{-\infty}^{\infty} u(z')\, G_i(r,a,z-z')\, dz' \qquad \text{(VIII.1)}$$

where the Green's function G_i is given in Eq. IV.2. Comparing this with Eq. IV.5b we see that the potentials ϕ_i defined, respectively, for the free-flooding and solid case differ as to the range of z' over which the integration is performed as well as to their Green's functions. As in the case of G_o, it is convenient to define separately the component of G_i, which is symmetrical about $z'=0$, and which alone contributes to the potential when the velocity distribution is similarly symmetrical:

$$g_i(r,a,z-z') = -\frac{1}{2\pi^2 a} \int_0^{\infty} \frac{J_0(k_r r)}{k_r J_1(k_r a)} \cos k_z z \cos k_z z' \, dk_z \qquad \text{(VIII.2)}$$

Assuming a constant velocity U over the radiating surface, this potential is again expressed in terms of an unknown velocity distribution $U\,\alpha(z')$:

$$\phi_i(r,z) = -4\pi a\, U\Big[\int_0^L g_i(r,a,z-z')\, dz' + \int_L^{\infty} \alpha(z') g_i(r,a,z-z')\, dz'\Big] \qquad \text{(VIII.3)}$$

The continuity condition at the boundary $r=a$, $z > L$, is again in the form of Eq. IV.9. The corresponding integral equation is therefore also, formally at least, similar to the integral equation derived for the solid cylinder, Eq. IV.10. Now,

however, the kernel $\Gamma(z-z')$ and the non-homogeneous term $H(z)$ are

$$\Gamma(z-z') = g_1(a,a,z-z') + g_o(a,a,z-z') \quad \text{(VIII.4a)}$$

$$H(z) = \int_0^L [g_1(a,a,z-z') + g_o(a,a,z-z')] \, dz' \quad \text{(VIII.4b)}$$

The expression for $\Gamma(z-z')$ can be simplified by using the Wronskian relation for H_o and J_o:

$$\Gamma(z-z') = \frac{1}{2\pi^2 a} \int_0^\infty \left[\frac{H_o(k_r a)}{k_r H_1(k_r a)} - \frac{J_o(k_r a)}{k_r J_1(k_r a)}\right] \cos k_z z \cos k_z z' \, dk_z$$

$$= \frac{i}{\pi^3 a^2} \int_0^\infty \frac{\cos k_z z \cos k_z z'}{k_r^2 J_1(k_r a) H_1(k_r a)} \, dk_z \quad \text{(VIII.5)}$$

The radiation impedance of the free-flooding shell differs from that of the solid cylinder in that the pressure on both the outer and inner surface contribute to it

$$Z = -\frac{4\pi a \, i\omega\rho}{U} \int_0^L [\Phi_o(a,z) - \Phi_1(a,z)] \, dz \quad \text{(VIII.6)}$$

The two components of this impedance stated in Eq. III.1 can again be separated. The impedance associated with Robey's mathematical model is:

$$Z_r = -(4\pi a)^2 \, i\omega\rho \int_0^L \int_0^L \Gamma(z-z') \, dz' \, dz$$

$$= -(4\pi a)^2 \, i\omega\rho \int_0^L H(z) \, dz \quad \text{(VIII.7)}$$

The correction term resulting from fluid flow across the cylindrical boundary $r=a$, $|z| > L$ is

$$Z_\alpha = -(4\pi a)^2 \, i\omega\rho \int_0^L \left[\int_L^\infty \alpha(z') \, \Gamma(z-z') \, dz'\right] dz \quad \text{(VIII.8)}$$

Since $\Gamma(z-z')$ is symmetrical in z and z', we can invert the order of integration and, using the definition of $H(z)$ in Eq. VIII.4b, write the unknown component of the radiation impedance as follows:

$$Z_\alpha = -(4\pi a)^2 \, i\omega\rho \int_L^\infty \alpha(z) H(z) \, dz \qquad \text{(VIII.9)}$$

The construction of the functional and the proof that it is stationary for the correct form of the unknown function $\alpha(z)$ parallels formally the proof given in Eqs. V.8 to 16 for the solid cylinder. The relation between the unknown impedance Z_α and the functional $J[\alpha]$, Eq. V.7, is also applicable. The variational solution of the free-flooding thin-walled shell is therefore formally identical with that of the solid cylinder provided the definitions of $\Gamma(z-z')$ and $H(z)$ given in Eqs. VIII.4a and b are used, instead of the corresponding definitions, Eq. IV.11 and 8, respectively, which apply to the solid cylinder case. We will see in the next section that this parallel does not hold when we assume a realistic free-flooding transducer or "squirter" whose wall thickness is not negligible.

The expression for the far field potential Φ_0 is still given by Eqs. VII.8 and 9. The expression for the potential Φ_1 is somewhat different, because of the contribution of the region $|z'| < L$ which is absent in the case of the solid cylinder:

$$\Phi_1(r,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(z') \int_{-\infty}^{\infty} \frac{J_0(k_r r)}{k_r J_1(k_r a)} \exp[ik_z(z-z')] \, dk_z \, dz' \qquad \text{(VIII.10)}$$

The integral over dk_z is given in Appendix A, Eq. A.14. The comments made in connection with Eqs. VII.1 and 2 apply.

IX. The Free-Flooding Cylindrical Transducer or "Squirter"

When we drop the assumption of a vanishing wall thickness, we must formulate the analysis in terms of three coaxial cylindrical boundaries and their respective radial velocities (Fig. 3).

(1) $r=a_i=a(1-\frac{h}{a})$; radial velocity $u_i(z)$

(2) $r=a$; radial velocity $u(z)$

(3) $r=a_o=a(1+\frac{h}{a})$; radial velocity $u_o(z)$ (IX.1)

Each of these cylindrical boundaries is of infinite extent in the z-direction.

The required modifications in the expressions for the potentials, Eqs. IV.5a and VIII.1, are self-evident and involve merely labeling a and u with the appropriate subscripts i or o. There is an equally obvious change in Eq. VIII.6 for the radiation impedance, where the potentials Φ_o and Φ_i must now be multiplied, respectively, by the ratios a_o/a and a_i/a. A non-trivial change must be introduced in the statement of the continuity condition, which now no longer takes the form $\Phi_i(a,z)=\Phi_o(a,z)$. Rather, we assume an incompressible potential in the annular region $a_i < r < a_o$. With this assumption the difference between the potentials Φ_i and Φ_o must be matched to the inertia force exerted by the fluid located in this annular region.

$$(1 + \frac{h}{a})\, \Phi_o(a_o,z) - (1 - \frac{h}{a})\, \Phi_i(a_i,z) = -2h\alpha(z)U, \quad \text{for } |z| > L \qquad (IX.2)$$

The assumption of an incompressible potential implies that the ratio $2h/\lambda$ is small. This condition is generally satisfied.*

We will now derive relations between the radial velocities on the three cylindrical surfaces defined above. As a result of the assumption of an incompressible potential in the annular region $a_i < r < a_o$ the velocities in the two semi-infinite regions prolonging the transducer can be derived from the requirement that inflow must balance outflow across the cylindrical boundaries r=a, a_i, and a_o

$$\left.\begin{aligned} u_i &= ua/a_i \\ &= u/(1-h/a) \\ u_o &= ua/a_o \\ &= u/(1 + h/a) \end{aligned}\right\} \text{for } |z| > L \qquad (IX.3)$$

*For the NRL magnetostrictive transducer ring $2h/\lambda$ is approximately 0.07. For transducers where this assumption is not valid, a compressible potential, Φ_a, must be constructed for the region $a_i < r < a_o$. The three potentials must satisfy two continuity conditions, viz: $\Phi_i(a_i)=\Phi_a(a_i)$ and $\Phi_a(a_o)=\Phi_o(a_o)$. Instead of one, two unknown radial velocity distribution over the boundaries, $r=a_i$ and $r=a_o$ must be determined from the two simultaneous integral equations arising from the two potential continuity requirements. These equations have been constructed, but it has not yet been verified whether a variational principle can be applied to their solution.

For $|z| < L$, i.e., in the region of the transducer, different relations must be used. The form of the equations relating these three velocities depends upon whether we are dealing with piezoelectric or magnetostrictive elements. In the case of piezoelectric transducers, the voltage applied to the electrodes located on the inner and outer surfaces of the ceramic ring produces a radial strain ϵ_r. The corresponding circumferential strain ϵ_φ of the mean surface of the element is the result of Poisson coupling:

$$\epsilon_\varphi = - \nu \epsilon_r \tag{IX.4}$$

where ν is Poisson's ratio. This circumferential strain is related to the radial velocity u of the mean surface as follows:

$$\epsilon_\varphi = u/(-i\omega a) \tag{IX.5}$$

Noting that ϵ_r is of opposite sign than ϵ_φ and hence than the displacement $u(-i\omega)^{-1}$, we find that the velocity of the outer and inner surface are respectively reduced and increased by the radial strain

$$u_i = u - h\dot{\epsilon}_r$$

$$u_o = u + h\dot{\epsilon}_r \tag{IX.6}$$

where a contraction corresponds to negative ϵ_r. Combining these equations we finally have

$$u_i = u(1 + \frac{h}{\nu a})$$

$$u_o = u(1 - \frac{h}{\nu a}) \tag{IX.7}$$

In the case of a ring-shaped magnetostrictive transducer the current in the solenoid produces a circumferential strain ϵ_φ. In this case it is the radial strain that results from Poisson coupling:

$$\epsilon_r = -\nu \epsilon_\varphi$$

$$= - \nu u(-i\omega a)^{-1} \tag{IX.8}$$

Combining these equations we now have

$$u_i = u(1 + \frac{\nu h}{a})$$

$$u_o = u(1 - \frac{\nu h}{a}) \tag{IX.9}$$

For the sake of brevity, we define the following coefficients, which tend to unity for small values of h/a:

$$\beta_{\substack{i\\o}} = \left(1 \mp \frac{h}{a}\right)\left(1 \pm \frac{h}{\nu a}\right), \quad \text{for piezoelectric transducers}$$

$$\beta_{\substack{i\\o}} = \left(1 \mp \frac{h}{a}\right)\left(1 \pm \frac{\nu h}{a}\right), \quad \text{for magnetostrictive transducers} \tag{IX.10}$$

If we now substitute these velocities in the expressions for the potentials used in the boundary condition, Eq. IX.2, we obtain a more complicated integral equation than for the two earlier configurations:

$$\frac{h}{2\pi a}\,\alpha(z) + \int_L^{\infty} \alpha(z')\left[\left(1+\frac{h}{a}\right)g_o(a_o,a_o,z-z') + \left(1-\frac{h}{a}\right)g_i(a_i,a_i,z-z')\right] dz'$$

$$= -\int_0^L \left[\beta_o\left(1+\frac{h}{a}\right)g_o(a_o,a_o,z-z') + \beta_i\left(1-\frac{h}{a}\right)g_i(a_i,a_i,z-z')\right] dz' \quad \text{for } |z| > L \tag{IX.11}$$

As will be seen shortly, the presence of the linear term in $\alpha(z)$ outside the integral sign, which makes this into a Fredholm integral equation of the second kind, does not interfere with the application of the variational principle. The presence of the coefficients β_i and β_o in the non-homogeneous term of the integral equation does unfortunately make the application of the variational principle impractical because the stationary potential $J[\alpha]$ which can be constructed, is no longer proportional to Z_α, as stated in Eq. V.7. To make this simple relation applicable, we must assume

$$\beta_i \approx \beta_o \approx 1 \tag{IX.12}$$

The error in this procedure is seen from Eq. IX.10 to be

$$e = \frac{h}{a}(1-\nu) - \left(\frac{h}{a}\right)^2 \nu \tag{IX.13a}$$

for magnetostrictive transducers. For piezoelectric transducers the error is larger

$$e = \frac{h}{a}\left(1-\frac{1}{\nu}\right) - \left(\frac{h}{a}\right)^2 \frac{1}{\nu} \tag{IX.13b}$$

The simple variational technique is therefore better suited to magnetostrictive than to piezoelectric transducers. For the magnetostrictive NRL transducer

ring ($2a=5\text{-}7/8$ in., $2h=\frac{1}{2}$ in.), the error computed from Eq. IX.13a is approximately 6 percent. It may seem inconsistent to introduce assumption IX.12 and not to drop the linear term in $\alpha(z)$ from Eq. IX.11, and the ratio h/a from terms of the form $(1 \pm h/a)$. Further work is necessary to determine whether retention of these terms increases accuracy. Until this is done, we shall retain these terms, because they do not complicate the variational technique. In addition to introducing the assumption stated in Eq. IX.11, we give a new definition of the function $\Gamma(z-z')$ and of $H(z)$

$$\Gamma(z-z') = (1 + \frac{h}{a})g_0(a_0,a_0,z-z') + (1 - \frac{h}{a})g_1(a_1,a_1,z-z') \qquad \text{(IX.14a)}$$

$$H(z) = \int_0^L [(1 + \frac{h}{a})g_0(a_a,a_0,z-z') + (1 - \frac{h}{a})g_1(a_1,a_1,z-z')]\, dz' \qquad \text{(IX.14b)}$$

Furthermore, we make the integral equation, Eq. IX.11, formally into a Fredholm equation of the first kind by the artifice of adding a Dirac delta function $\delta(z-z')$ to the kernel:

$$\int_L^\infty [\Gamma(z-z') + \frac{h}{2a\pi}\,\delta(z-z')]\,\alpha(z')\, dz' = -\,H(z), \quad \text{for } |z| > L \qquad \text{(IX.15)}$$

The functional $J[\alpha]$ is still of the same form as in the two earlier analyses, Eq. V.9, and the definition of the functional $A[\alpha]$, also remains formally the same, Eq. V.10a. The functional $B[\alpha]$ is, however, different.

$$B[\alpha] = \int_L^\infty \int_L^\infty \alpha(z)[\Gamma(z-z') + \frac{h}{2a\pi}\,\delta(z-z')]\,\alpha(z')\, dz\, dz' \qquad \text{(IX.16)}$$

We will now show that the stationary character of $J[\alpha]$ can be established as before. Setting the increment of the functional equal to 0, we have

$$\delta J[\alpha] = 0 = \frac{2A[\alpha]}{B[\alpha]} \int_L^\infty H(z)\,\delta\alpha(z)\, dz\, -$$

$$-\,\frac{A^2[\alpha]}{B^2[\alpha]} \int_L^\infty \int_L^\infty [\Gamma(z-z') + \frac{h}{2a\pi}\delta(z-z')][\alpha(z)\delta\alpha(z') + \alpha(z')\delta\alpha(z)]dz\, dz' \qquad \text{(IX.17)}$$

The order of integration of the $\alpha(z)\delta\alpha(z')$ term in the double integral can be inverted.

$$\int_L^\infty\int_L^\infty = 2\int_L^\infty\int_L^\infty [\Gamma(z-z') + \frac{h}{2a\pi}\,\delta(z-z')]\,\alpha(z')\,\delta\alpha(z)\,dz\,dz' \qquad (IX.18)$$

From here on, the proof parallels exactly the steps from Eqs. V.13 to 16 and will therefore not be repeated. The radiation impedance correction factor Z_α, is once again formally given by Eq. VIII.8, with $\Gamma(z-z')$ defined in Eq. IX.14a. Thus by setting the coefficients β equal to unity, Z_α can still be expressed in terms of the functional $J[\alpha]$ as in Eq. V.7. The component of the radiation impedance associated with Robey's mathematical model is

$$Z_r = -i\omega\rho(4\pi a)^2\int_0^L\int_0^L [\beta_o(1+\frac{h}{a})g_o(a_o,a_o,z-z')+\beta_i(1-\frac{h}{a})g_i(a_i,a_i,z-z')]\,dz'\,dz \qquad (IX.19)$$

Even if the coefficients β had not been set equal to unity, a stationary potential $J[\alpha]$ could have been constructed with

$$A[\alpha] = \int_L^\infty \alpha(z)\{\int_0^L [\beta_o(1+\frac{h}{a})g_o(a_o,a_o,z-z')+\beta_i(1-\frac{h}{a})g_i(a_i,a_i,z-z')]\,dz'\}dz \qquad (IX.20)$$

instead of the expression in Eq. V.10a. The usefulness of the variational method is however impaired, because the radiation impedance component Z_α does not change in the same manner as $A[\alpha]$: Z_α is given, as before, by Eq. VIII.8 with $\Gamma(z-z')$ as defined in Eq. IX.14a. It therefore does not involve the coefficients β, whether we set the coefficients equal to unity or not. Z_α is therefore not proportional to $A[\alpha]$, Eq. IX.20, and Eq. V.7 relating Z_α and $J[\alpha]$ does not apply. Thus even though the variational method can be used for "squirters" whose walls are too thick to permit setting the coefficients β equal to unity, the unknown coefficients in the trial function $\alpha(z)$ must be solved for before computing Z_α. Whether such a procedure is competitive with the non-variational solutions presented in Appendix B for thick-walled "squirters" can best be verified empirically after numerical calculations have been performed.

We shall now extend the variational technique to arbitrary non-axisymmetric velocity distributions of the radiating surface.

X. Cylinders Vibrating in Longitudinal and in Non-Axisymmetric Modes

The analysis of the "squirter" and of the solid cylinder can be directly adapted to the case of nonuniform axisymmetric velocity distributions over the radiating surface, by replacing the constant velocity U in the region $|z| < L$ with a z-dependent velocity $u(z)$. If $u(z)$ is not symmetrical about $z=0$, the most convenient approach is to consider the velocity distribution as the sum of a symmetrical distribution u_s and of an antisymmetric distribution u_a. As we are dealing with a linear problem, we can add the corresponding potentials. We first compute the potential associated with u_s as in the preceding analysis, setting $\alpha_s(z)=\alpha_s(-z)$, and using the corresponding Green's functions g_o, Eq. IV.6, and g_i, Eq. VIII.2 (the latter in the case of the open-ended cylinder). To this we add the potential resulting from the velocity distribution u_a for which $\alpha_a(z) = -\alpha_a(-z)$. The suitable partial Green's functions are obtained by modifying the expressions for g_o and g_i given, respectively, in Eqs. IV.6 and VIII.2, sin $k_z z$ sin $k_z z'$ being substituted for the product of cosines. We must thus solve two uncoupled integral equations for the two unknown velocity distributions, α_s and α_a.* Unless we proceed in this fashion, the solid cylinder with arbitrary velocity distribution $u(z)$ gives rise to three different potentials Φ_o, Φ_{i+}, Φ_{i-} which in turn result in two distinct boundary conditions corresponding, respectively, to the regions $z < L$ and $z > L$. The two resulting integral equations will thus be coupled, each involving both unknown velocity distributions, $\alpha(z < 0)$ and $\alpha(z > 0)$.

In the case of a piston or ring vibrating in phase on a finite cylindrical baffle or array, the velocity distribution of the active element is of course constant and hence symmetrical over the midplane ($z=0$) of the element, but unless this element is centrally located with respect to the baffle or array, the velocity distribution $\alpha(z)$ will not be symmetrical. In this respect, the present mathematical model differs from Robey's model, in which the potential and the velocity distribution are always symmetrical about the plane of symmetry of the active element.

A configuration of practical interest is that of a solid cylinder whose end caps reciprocate in the axial direction. This situation arises as a result of Poisson coupling with predominantly radial, axisymmetric modes. End cap motion can contribute the major portion of the sound field in the case of the so-called

*This procedure will be illustrated in the report dealing with an array of ring transducers (see footnote on p. 1).

accordion modes, which are predominantly longitudinal. To account for an axisymmetric velocity distribution $v(r')$ over the end caps, the following integral is added to the surface integrals in Eq. IV.5b:

$$\Delta\Phi_{i\pm}(r,z) = \mp 4\pi \int_0^a v(r')\, G_i(r,r',z \mp L)\, r'\, dr' \qquad (X.1)$$

where v has been taken positive in the positive z-direction. If the cylindrical surface of the radiator is motionless, the potential Φ_o, in the region $r > a$ is associated entirely with the velocity distribution $\alpha(z)$ across the two surfaces $(r=a,\ z \gtrless L)$.

The variational analysis can be extended further, to include an arbitrary non-axisymmetric velocity distribution

$$u(\varphi,z) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} U_m f_m(z) \exp(im\varphi), \quad \text{for } |z| < L \qquad (X.2)$$

U_m is a modal velocity amplitude, the maximum value of the function $f_m(z)$ being unity. To each Fourier component U_m of the velocity, corresponds a partial potential $\Phi_m(r,z)$, the total potential being of the form

$$\Phi(r,z) = \sum_{m=-\infty}^{\infty} \Phi_m(r,z) \exp(im\varphi) \qquad (X.3)$$

The partial potentials are obtained from Eqs. IV.5 and Eq. VIII.1, by substituting in place of the axisymmetric Green's functions given in Eqs. IV.1 and 2 the following.

$$G_{om}(r,z,\varphi-\varphi',z-z') = -\frac{1}{4\pi^2 a} \exp[im(\varphi-\varphi')] \int_{-\infty}^{\infty} \frac{H_m(k_r r)}{k_r H_m'(k_r a)} \exp[ik_z(z-z')]dk_z, \quad \text{for } r > a \qquad (X.4)$$

The near field value of this integral has been evaluated by Greenspon and Sherman.[6] Its asymptotic far field value is given by Laird and Cohen:[5]

$$G_{om}(R,\theta,\varphi-\varphi')\Big|_{r'=a} = \frac{i \exp(ikR)}{2\pi^2 ak\, R \sin\theta} \frac{\exp[im(\varphi-\varphi'+\pi/2)]}{H_m'(ka \sin\theta)}, \quad \text{for large } R \qquad (X.5)$$

The corresponding Green's function in the cylindrical region is derived in Appendix A, Eq. A.6:

$$G_{im}(r,a,z-z',\varphi-\varphi') = \frac{1}{4\pi^2 a}\exp[im(\varphi-\varphi')]\int_{-\infty}^{\infty}\frac{J_m(k_r r)}{k_r J'_m(k_r a)}\exp[ik_z(z-z')]\,dk_z, \quad \text{for } r<a \qquad (X.6)$$

The integral is evaluated in Eq. A.18. Each potential ϕ_m is the sum of two components: a potential ϕ_{mr} associated with the known modal velocity distribution $U_m f_m(z)$ over the radiating surface, and a component $\phi_{m\alpha}$ associated with the unknown modal velocity distribution $U_m \alpha_m(z)$ in the two regions $|z| > L$. The former component ϕ_{mr} is of course the component computed from Robey's mathematical model of the cylinder prolonged by two semi-infinite cylindrical baffles. The modal impedance associated with the mth mode of the radiator can be expressed, as in Eq. III.1, as the sum of Robey's impedance, associated with ϕ_{mr}, and of a correction term associated with $\phi_{m\alpha}$:

$$Z_{mr} = -\frac{i\omega\rho a\pi}{U_m}\int_{-L}^{L}\phi_{mr}(a,z)f_m(z)\,dz$$

$$Z_{m\alpha} = -\frac{i\omega\rho a\pi}{U_m}\int_{-L}^{L}\phi_{m\alpha}(a,z)f_m(z)\,dz \qquad (X.7)$$

This impedance can be used to compute the generalized force associated with radiation loading of the mth elastic mode of the cylinder, and hence the modal impedances and natural frequencies of the submerged cylinder. Modal radiation impedances can also be combined to compute the self-radiation impedance of rigid pistons in <u>finite</u> cylindrical baffles.* Because of the similarity in the form of the Green's functions of the axisymmetric case analyzed in detail in this report and of the non-axisymmetric radiator configurations, it is obvious that the integral equations which $\alpha_m(z)$ must satisfy are of the same form as the integral equations which define $\alpha(z)$ in the axisymmetric case. Functionals $J_m[\alpha_m]$ stationary with respect to the correct function $\alpha_m(z)$ can be constructed and are found to be of the same form as the functional $J[\alpha]$ constructed earlier for the axisymmetric radiator. The proof that the impedance $Z_{m\alpha}$ is proportional to $J_m[\alpha_m]$ for the

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*The component Z_{mr} of this impedance is computed by Greenspon and Sherman.[8]

correct form of $C_m(z)$ parallels the proof in Section V. The variational technique illustrated in Section VI can therefore be used to evaluate $Z_{m\alpha'}$, and need not be repeated here.

In the case of the solid cylinder, nonrigid vibrating end caps can be accounted for by adding to the expression for the potentials $\Phi_{i\pm}$ a surface integral over the end caps:

$$\Delta\Phi_{i\pm} = \mp 2 \sum_{m=-\infty}^{\infty} \int_0^{2\pi} \int_0^{a} v(r',\varphi')\, G_{im}(r,r',\varphi-\varphi',z \mp L)\, r'\, dr'\, d\varphi' \qquad (X.8)$$

Appendix A*

DERIVATION AND EVALUATION OF THE GREEN'S FUNCTION G_i FOR THE CYLINDRICAL REGION $r < a$

1. Construction of the Green's Function

This derivation parallels the construction of the Green's function G_o for the region $r > a$ given by Robey[6] for Neumann boundary conditions ($\partial\Phi/\partial n$ known, $\partial G/\partial n$ made to vanish on boundary) and by Papas[18] for Dirichlet boundary conditions (Φ known, G made to vanish on boundary). The Green's function associated with the mth Fourier component of the velocity distribution in φ can be expressed in terms of an inverse Fourier transform in $(z-z')$:

$$G_{im}(r,r',\varphi-\varphi',z-z') = \frac{\exp[im(\varphi-\varphi')]}{4\pi^2} \int_{-\infty}^{\infty} G_{im}(r,r',k_z)\exp[ik_z(z-z')]\, dk_z \qquad (A.1)$$

The Green's function satisfies, by definition, the non-homogeneous Helmholtz equation expressed in cylindrical coordinates. Consequently, the transform of the Green's function satisfies the following equation:

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + k^2 - k_z^2 - \frac{m^2}{r^2}\right] G_m(r,r',k_z) = -\frac{\delta(r-r')}{r} \qquad (A.2)$$

The steps leading from the non-homogeneous Helmholtz equation to Eq. A.2 are presented in detail in reference 11. Except when $r=r'$, Eq. A.2 is of the form of Bessel's equation. A suitable solution to this equation must be regular when $r_<$ vanishes ($r_<$ and $r_>$ are, respectively, the smaller and the larger of the quantities r and r'). The solution of Eq. A.2 must therefore contain only Bessel functions of argument proportional to $r_<$; Neumann or Hankel functions can only have arguments proportional to $r_>$ or a. A combination of cylinder functions which satisfies these conditions, and whose radial derivative $\partial G/\partial r'$ vanishes on

*This material is included here because it does not appear to be available in the literature. Appendix A is condensed from a Harvard Acoustics Research Laboratory Memorandum.[17] Analytical details and proofs which had to be omitted to keep the length of this report within reason, can be found in reference 17.

the cylindrical boundary $r > = r'$, is of the form:

$$G_{im}(r,r',k_z) = A_m J_m(k_r r <)[H_m'(k_r a) J_m(k_r r >) - H_m(k_r r >) J_m'(k_r a)] \tag{A.3}$$

The coefficients A_m are determined from the equation defining the discontinuity of the first derivative of the Green's function:

$$r \frac{\partial}{\partial r} G_{im}(r,r',k_z) \Bigg|_{r=r'-}^{r=r'+} = -1 \tag{A.4}$$

After some transformations the coefficient A_m is found to be

$$A_m = \frac{-\pi i}{2J_m'(k_r a)} \tag{A.5}$$

When we substitute this coefficient in Eq. A.3, set $r' = r > = a$ and $r = r <$, and use the Wronskian relation between $H_m(k_r a)$ and $J_m(k_r a)$, we finally obtain the following expression for the Green's function

$$G_{im}(r,a,\varphi-\varphi',z-z') = \frac{\exp[im(\varphi-\varphi')]}{4\pi^2 a} \int_{-\infty}^{\infty} \frac{J_m(k_r r)}{k_r J_m'(k_r a)} \exp[ik_z(z-z')]\, dk_z \tag{A.6}$$

For the axisymmetric case, we have $J_m'=J_0'=-J_1$, which yields the Green's function given in Eq. IV.2. We will follow the notation used in the body of the report, whereby the subscript m is omitted when m=0, i.e., in what follows, G_i, I, R_n and k_n indicate, respectively, G_{i0}, I_0, R_{0n} and k_{0n}.

2. Evaluation of the Inverse Fourier Transform

We will not evaluate the infinite integral in Eq. A.6. For the purpose of analysis the wave number is assumed to have a small imaginary component

$$k_z = \xi + i\eta \tag{A.7}$$

The integration in Eq. A.6 will be performed in the complex plane along a closed counterclockwise contour including the real axis and a half circle of infinite radius (Fig. 4). By the residue theorem,[19] the value of the contour integral is $2\pi i$ times the sum of the n residues at the poles k_{mn} of the integrand.

$$\int_{-\infty}^{\infty} I_m dk_z = 2\pi i \sum_n R_{mn+} \text{, for } z-z' > 0\text{, and } k_{mn} \text{ in upper half-plane}$$

$$= -2\pi i \sum_n R_{mn-} \text{, for } z-z' < 0\text{, and } k_{mn} \text{ in lower half-plane} \qquad \text{(A.8)}$$

where I_m stands for the integrand in Eq. A.6. This contour integral equals the integral along the real axis if the contribution of the half-circle vanishes. To achieve this condition the integrand must vanish as k_z tends to infinity, i.e., $|\exp[ik_z(z-z')]|$ in the integrand must decrease exponentially with increasing k_z. Hence

$$\eta > 0\text{, for } z-z' > 0 \quad \text{(Integration in upper half-plane)}$$

$$\eta < 0\text{, for } z-z' < 0 \quad \text{(Integration in lower half-plane)} \qquad \text{(A.9)}$$

Like k_z, k can be assumed to be complex. Its infinitesimal imaginary component can be associated with viscous losses in the acoustic medium, if a physical interpretation is desired. The complex quantity +k lies just above the real axis, and -k just below it. The contours of integration are then as shown in Fig. 4.

We will first evaluate the axisymmetric Green's function. The integrand I has poles at $k_r=0$, i.e., at $k_z = \pm k$. Taking the asymptotic expression of the Bessel functions J_0 and J_0' as their argument tends to zero, we find that the integrand tends to

$$I(k_z) \rightarrow \frac{2\exp[ik_z(k_z-z')]}{a^2(k_z+k)(k_z-k)}\text{, as } k_z \rightarrow \pm k \qquad \text{(A.10)}$$

The two simple poles at $k_z = \pm k$ give rise to the following residues

$$R_{0+} = \frac{\exp[ik(z-z')]}{ka^2}\text{, for } z-z' > 0$$

$$R_{0-} = \frac{-\exp[-ik(z-z')]}{ka^2}\text{, for } z-z' < 0 \qquad \text{(A.11)}$$

Other poles, all of them simple, occur at the roots $(k_n a)$ of $J_0'(k_r a)$. The corresponding residues are

$$R_{n\pm} = \frac{\pm 2\exp[\pm i(k^2-k_n^2)^{\frac{1}{2}}(z-z')]}{a^2(k^2-k_n^2)^{\frac{1}{2}}} \frac{J_0(k_n r)}{J_0(k_n a)-J_2(k_n a)} \qquad \text{(A.12)}$$

Because

$$R_{n+}(z-z') = -R_{n-}(z'-z) \;, \tag{A.13}$$

the integral can be stated as follows, without regard for the relative magnitude of z and z':

$$\int_{-\infty}^{\infty} I\, dk_z = \frac{2\pi i}{a}\left[\frac{\exp(ik|z-z'|)}{ka} + 2\sum_{n=1}^{\infty}\frac{\exp[i(k^2-k_n^2)|z-z'|]}{(k^2-k_n^2)^{\frac{1}{2}}\, a}\,\frac{J_0(k_n r)}{J_0(k_n a)-J_2(k_n a)}\right] \tag{A.14}$$

For all roots $k_n a$ of J_0' which exceed ka, the terms under the summation sign decay exponentially with increasing $|z-z'|$. For higher order roots, the terms under the summation sign are proportional to

$$\frac{\exp(-n\pi|z-z'|/a)}{n(ra)^{\frac{1}{2}}}, \quad \text{for n large} \tag{A.15}$$

The series expression in Eq. A.14 is thus seen to be convergent except for $z=z'$, which fulfills the requirement of a Green's function.

The convergence of the Green's function as $|z|$ tends to infinity is not spherical, but relies on the small imaginary component of the wave number k. The reason is that we have constructed a Green's function suitable for cylindrical region, viz. a circular pipe, where only viscous losses, but no spreading losses occur. The potential ϕ_i does, however, vary as $|z|^{-1}\exp(ikz)$ for large $|z|$, because the radial velocity $\alpha(z)$ gives rise to a net outflow of acoustic energy from the region $r < a$ (see Eqs. VI.1 to 6, and comments following Eq. VII.2).

We now turn to the evaluation of the non-axisymmetric Green's function. At $k_z = \pm k$, for $m > 0$ the integrand tends to

$$I_m \to \frac{(r/a)^m}{m}\exp[ik_z(z-z')] \qquad \text{as } k_z \to \pm k \tag{A.16}$$

There are therefore no poles at $k_z = \pm k$, only for $m=0$. For $m\neq 0$, all the residues are associated with the roots of J_m':

$$R_{mn\pm} = \pm\frac{4\exp[\pm i(k^2-k_{mn}^2)^{\frac{1}{2}}(z-z')]}{a^2(k^2-k_{mn}^2)^{\frac{1}{2}}}\,\frac{J_m(k_{mn}r)}{2J_m(k_{mn}a)-J_{m-2}(k_{mn}a)-J_{m+2}(k_{mn}a)} \quad \text{for } m\neq 0 \tag{A.17}$$

Once again the integrand can be expressed without regard for the sign of $(z-z')$:

$$\int_{-\infty}^{\infty} I_m \, dk_z = \frac{8\pi i}{a^2} \sum_{n=1}^{\infty} \frac{\exp[i(k^2-k_{mn}^2)^{\frac{1}{2}}|z-z'|]}{(k^2-k_{mn}^2)^{\frac{1}{2}}} \frac{J_m(k_{mn}r)}{2J_m(k_{mn}a)-J_{m-2}(k_{mn}a)-J_{m+2}(k_{mn}a)}$$

for $m \neq 0$ (A.18)

The higher order terms are again found to be proportional to the expression in Eq. A.15, and thus to conform to Green's function requirement by converging for $z \neq z'$. A proof was given in ref. 17 of the fact that even though the function $(k^2-k_z^2)^{\frac{1}{2}}$ has a branch cut in the region $|k_z| < k$ of the real axis, the integrand in Eq. A.6 does not have branch points at any of its poles.

Appendix B

NON-VARIATIONAL TECHNIQUES FOR SOLVING THE "SQUIRTER" INTEGRAL EQUATION, EQ. IX.11

In Section IX it was shown that the variational technique developed for the solid cylinder is applicable to the free-flooding cylinder only when the coefficients β_i and β_o, Eq. IX.10, can be set equal to unity, i.e., when the ratio of wall thickness to radius $2h/a$ is small. It was also pointed out that when the ratio of wall thickness to acoustic wavelength $2h/\lambda$ is not small enough to make the compressibility of the fluid annulus in the region $a_i < r < a_o$ negligible, a complicated analysis involving two coupled simultaneous integral equations must be used. The purpose of this Appendix is to present a technique for dealing with a "squirter" for which the ratio $2h/a$ is not small enough to permit setting the coefficients β in Eq. IX.10 equal to unity, even though the corresponding ratio $2h/\lambda$ is sufficiently small to allow us to ignore the compressibility of the fluid in the annular region. The most straightforward approach is to evaluate the unperturbed potentials, which are obtained by setting $\alpha(z) = 0$, and to use these potentials to compute a perturbation solution of $\alpha(z)$ from Eq. IX.2:

$$\alpha^{(0)}(z) = -\frac{2\pi a}{h}\int_0^L [\beta_o(1+\tfrac{h}{a})g_o(a_o,a_o,z-z') + \beta_i(1-\tfrac{h}{a})g_i(a_i,a_i,z-z')]\,dz' \qquad (B.1)$$

The perturbation solution of the impedance correction factor Z_α is obtained by substituting this expression for $\alpha(z)$ in Eq. VIII.8, with $\Gamma(z-z')$ as defined in Eq. IX.14. The far field potentials can of course also be obtained in a straightforward fashion by substituting $\alpha^{(0)}(z)$ in Eq. VII.8 and 9, for Φ_o, and Eq. VIII.3 for Φ_i.

The perturbation solution can be improved by iteration as follows: One substitutes $\alpha^{(0)}(z')$ for $\alpha(z')$ in the z'-integral in Eq. IX.11 and solves for $\alpha(z)$. This amounts to solving Eq. IX.1 for $\alpha(z)$ using the perturbation solutions of Φ_o and Φ_i. If this iteration process is repeated p times, one finds that the pth

iterate of $\alpha(z)$ is related to the (p-1)th iterate as follows:

$$\alpha^{(p)}(z) = -\frac{2\pi a}{h}\left\{\int_0^L [\beta_o(1+\frac{h}{a})g_o(a_o,a_o,z-z') + \beta_1(1-\frac{h}{a})g_1(a_1,a_1,z-z')]\,dz' + \int_L^\infty \alpha^{(p-1)}(z')\,\Gamma(z-z')\,dz'\right\} \tag{B.2}$$

where $\Gamma(z-z')$ is defined in Eq. IX.14a.

We will now present a finite-difference procedure for solving the integral equations. Instead of requiring that the integral equation, Eq. IX.11, be satisfied for all values of z in the region $|z| > L$, we satisfy it at a finite number of points $z_n = z_1, z_2, z_3 \ldots z_N$ separated by intervals $2d_n$. These intervals should be selected smaller in regions close to the transducer extremities, which make a more important contribution to the potentials than more distant regions. Furthermore, we assume that the unknown function $\alpha(z)$ has a constant value α_n in each interval $(z_n-d_n) < z < (z_n+d)$ and varies discontinuously from one interval to the next. We thus arrive at a set of N simultaneous equations in N unknown quantities α_n:

$$\begin{bmatrix} \frac{h}{2\pi a} + 2d_1\Gamma(0) & 2d_2\Gamma(z_1-z_2) \ldots & 2d_N\Gamma(z_1-z_N) \\ 2d_1\Gamma(z_2-z_1) & \frac{h}{2\pi a} + 2d_2\Gamma(0) \ldots & \ldots \\ \ldots & \ldots & \ldots \\ 2d_1\Gamma(z_N-z_1) & \ldots & \frac{h}{2\pi a} + 2d_N\Gamma(0) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_N \end{Bmatrix} = - \begin{Bmatrix} F(z_1) \\ F(z_2) \\ \cdot \\ F(z_N) \end{Bmatrix}$$

where

$$F(z_n) = \int_0^L [\beta_o(1+\frac{h}{a})g_o(a_o,a_o,z_n-z') + \beta_1(1-\frac{h}{a})g_1(a_1,a_1,z_n-z')]\,dz' \tag{B.3}$$

The two Green's functions which enter into the linear combination $\Gamma(z-z')$, Eq. IX.14a, have a pole at $z=z'$. The diagonal terms in the above matrix do not, however, display a singularity, since they are equivalent to an integral of $\Gamma(z-z')$ over z', which, like the potential, is well behaved.

Solving this set of equations for the values of α_n we can compute the radiation impedance component Z_α from Eq. VIII.8

$$Z_\alpha = -2(4\pi a)^2 \, i\omega\rho \sum_{n=1}^{N} d_n \alpha_n \int_0^L \Gamma(z-z_n)\, dz \qquad (B.4)$$

Robey's impedance component, Z_r, is given in Eq. IX.19.

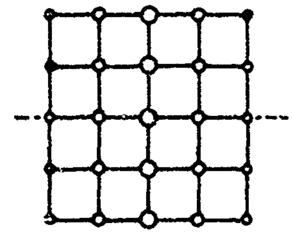
When applying the finite-difference method to the non-axisymmetric velocity distributions discussed in Section X, it is not necessary to construct a two-dimensional grid of points (z_n, φ_p) over the two cylindrical surfaces $r=a, |z| > L$. Rather, a one-dimensional set of finite-difference equations in z, of the form of Eq. B.3 applies to each modal velocity distribution $\alpha_m(z)$ associated with the non-axisymmetric Green's functions, Eq. X.3 and 5.

In conclusion, it is recalled (end of Section IX) that a variational solution is applicable, even when 2h/a is not small, but that Z_α cannot be obtained directly from the functional $J[\alpha]$, without also solving for the unknown coefficients in the trial function $\alpha(z)$.

(a) <u>Finite-Difference Calculation</u>

(Baron, Matthews and Bleich,[2] Chen and Schweikert[3])

Grid of point-sources approximates radiating surface. Source strength determined from finite-difference solution of Helmholtz integral equation



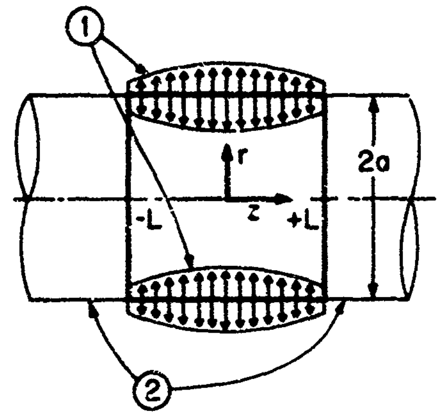
(b) <u>Robey's Mathematical Model of the Cylindrical Source</u>

(Laird and Cohen,[5] Robey,[6] Greenspon,[7,8] and Sherman[9])

Integral equation circumvented by assuming rigid cylindrical baffles r=a, |z| > L, and by constructing Green's function for which $(\partial G/\partial r')=0$ for $r'=a$

1) Known velocity distribution over radiating surface

2) Semi-infinite rigid baffles



(c) <u>Robey's Mathematical Model of "Squirter"</u>

(Robey[6])

Same as Fig. 1(b) but plane baffles in regions $r > a$, $z = \pm L$

1) Infinite rigid baffles

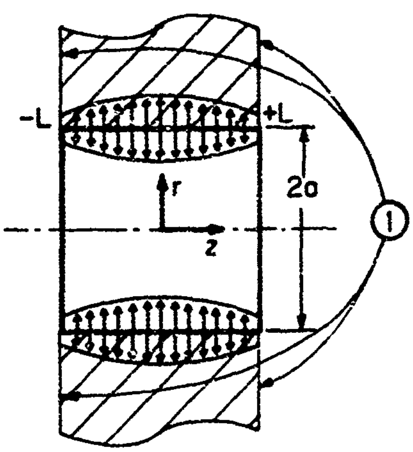


Fig. 1. REVIEW OF PUBLISHED ANALYSES OF CYLINDRICAL RADIATORS (See Table 2 on p. 3)

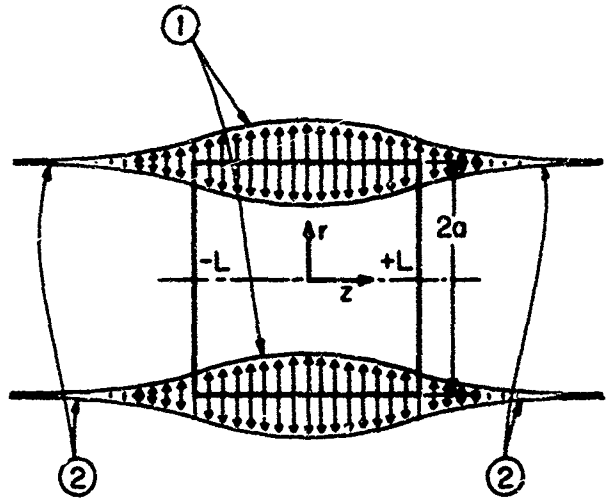


Fig. 2. SUMMARY OF PRESENT APPROACH (See Table 2 on p. 3)

1) Known velocity distribution of the radiating surface ($-L < z < L$, $r=a$)

2) Unknown velocity distribution $\alpha(z)$ satisfies integral equation on surfaces $r=a$, $|z| > L$ [z-dependent phase shift of $\alpha(z)$ is not indicated]

Radiation impedance $= Z_r + Z_\alpha$

where Z_r = impedance computed from Robey's mathematical model, Fig. 1b

Z_α = correction associated with unknown velocity distribution $\alpha(z)$

Variational principle: for correct $\alpha(z)$, $Z_\alpha \propto J[\alpha]$

$$\frac{\delta J[\alpha]}{\delta \alpha} = 0$$

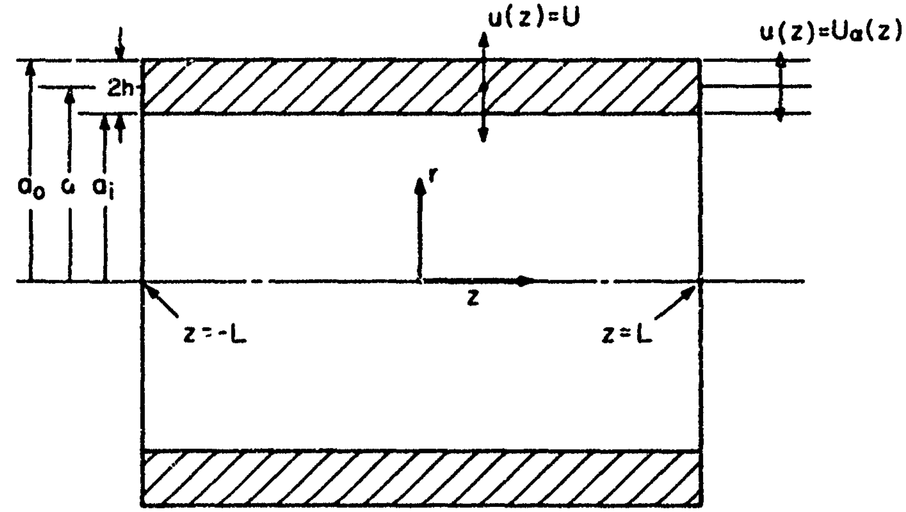


Fig. 3 GEOMETRY OF "SQUIRTER"

$u(z)$ = radial velocity on cylindrical surface $r=a$

$u(z) = U$ for $-L < z < L$

$u(z) = U\alpha(z)$ for $z < -L$ and $z > L$

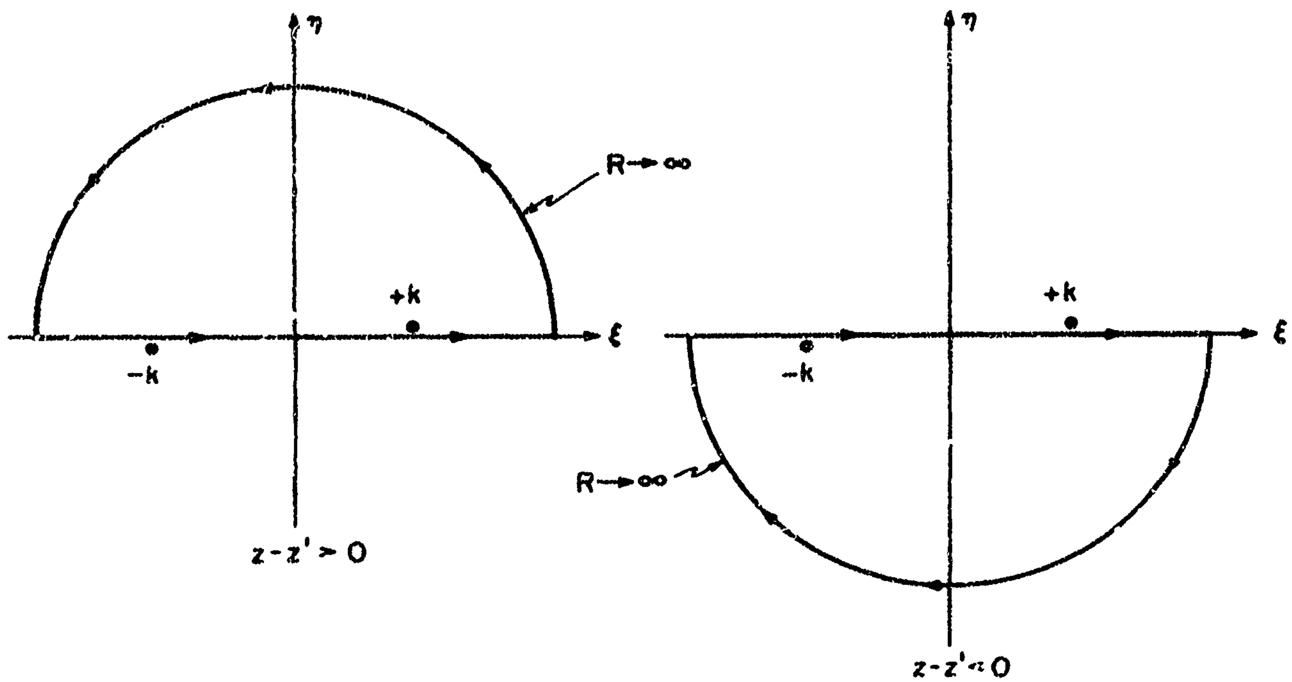


Fig. 4. CONTOUR INTEGRATION OF THE INVERSE TRANSFORM OF THE GREEN'S FUNCTION G_1 (Eqs. IV.2 and A.6)

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(Task NR 185-301, Contract Nonr-2739(00)
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Finite length effects in solid cylindrical sound sources and in free-flooding "squirters" are evaluated by means of a variational technique which parallels the Levine-Schwinger variational principle for diffraction problems. The radiation impedance is expressed as the sum of two components: (1) The impedance computed from Robey's mathematical model which has been generally applied to cylindrical transducers, but which does not account for end effects; (2) the impedance associated with the radial velocity distribution over the two semi-infinite cylindrical surfaces extending the radiating surface. This technique is also applied to longitudinal modes of the solid cylinder, and to non-axisymmetric model configurations. A subsequent report (CAA U-178-48) extends this analysis to arrays of coaxial free-flooding ring-transducers.

1. Physics - acoustics
2. Sonar transducers - radiation loading
I. M. C. Junger

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Cambridge, Massachusetts. Report U-177-48
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1 March 1964, 48 pp. 4 figs. refs.
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